Preliminary work.

This book assumes that you are already familiar with a number of mathematical ideas from your mathematical studies in earlier years.

This section outlines the ideas which are of particular relevance to Unit Two of the *Mathematics Methods* course and for which familiarity will be assumed, or for which the brief explanation given here may be sufficient to bring your understanding of the concept up to the necessary level.

Read this "preliminary work" section and if anything is not familiar to you, and you don't understand the brief mention or explanation given here, you may need to do some further reading to bring your understanding of those concepts up to an appropriate level for this unit. (If you do understand the work but feel somewhat "rusty" with regards to applying the ideas some of the chapters afford further opportunities for revision as do some of the questions in the miscellaneous exercises at the end of chapters.)

Note especially the last inclusion in this section which involves factorising expressions of the form $a^n - b^n$. Other than that of $a^2 - b^2$ the reader is probably not familiar with the factorisations given here but by reading the brief explanation given in the last section of this Preliminary work the reader should bring their understanding of the concept up to the required understanding for its later use in this text.

☞ Chapters in this book will continue some of the topics from this preliminary work by building on the assumed familiarity with the work.

☞ The miscellaneous exercises that feature at the end of each chapter may include questions requiring an understanding of the topics briefly explained here.

- **Number:**
  It is assumed that you are already familiar with counting numbers, whole numbers, integers, factors, multiples, prime numbers, composite numbers, square numbers, negative numbers, fractions, decimals, the rule of order, percentages, the square root and the cube root of a number, powers of numbers (including zero and negative powers), and can use this familiarity appropriately. An ability to simplify simple expressions involving square roots is also assumed.

  e.g. $\sqrt{8} = \sqrt{4 \times 2} = 2\sqrt{2}$ $\qquad \sqrt{27} + \sqrt{75} = \sqrt{9 \times 3} + \sqrt{25 \times 3} = 3\sqrt{3} + 5\sqrt{3} = 8\sqrt{3}$

  An understanding of numbers expressed in *standard form* or *scientific notation*, e.g. writing 260 000 in the form $2{\cdot}6 \times 10^5$ or writing 0·0015 in the form $1{\cdot}5 \times 10^{-3}$, is also assumed.

- **Percentages:**
  It is assumed you are familiar and comfortable with the use of percentages and in particular their use in the concepts of **simple interest** and **compound interest**.

- **Rounding:**

  Answers to some calculations may need rounding to a suitable or specified accuracy. For example, if we were to cut a 5 metre length of string into seven equal pieces it would be unwise to claim that each piece would be of length

  0·7142857143 metres

  the answer a calculator might give when asked to calculate 5 ÷ 7. Not only is this answer too accurate for the task involved, it is also ludicrous to claim such accuracy when the information used to calculate it, i.e. the length being 5 metres, would not have been measured to this accuracy itself.
  Instead we might round the answer to perhaps 2 decimal places. I.e. 0·71 metres.

  In some cases the situation may make *truncating* more appropriate than rounding. Suppose for example we have $10 and wish to buy as many chocolate bars costing $2·15 each as possible. Whilst $10 ÷ $2·15 is 4·65 if we round to two decimal places, 4·7 if we round to one decimal place and 5 if we round to the nearest integer, a more appropriate answer is obtained by truncating to 4 as that is the number of chocolate bars we would be able to buy with our $10 (and we would have $1·40 change). If we truncate to an integer we discard the decimal part entirely.

  Suppose the estimated cost to a company of manufacturing 18 models of a new machine is $375 000. Dividing this amount by 18 gives an estimated cost of

  $20 833·33 each (nearest cent).

  However, this is again likely to be too accurate for the situation and we might instead round to the nearest $100, in this case $20 800. Alternatively we could say that we have rounded to three *significant figures*. To round to a number of significant figures we count that number of digits and then use the next digit to apply our usual rounding rules.

For example	526 086·9565	is	500 000	to 1 significant figure	
		is	530 000	to 2 significant figures	
		is	526 000	to 3 significant figures	
		is	526 100	to 4 significant figures	etc.

  For very small numbers we do not count any initial zeros as significant figures.

For example	0·000 310 486 2	is	0·000 3	to 1 significant figure	
			0·000 31	to 2 significant figures	
			0·000 310	to 3 significant figures	
			0·000 310 5	to 4 significant figures	etc.

- **Indices.**

  Whilst it is anticipated that you are familiar with the idea of

  raising a number to some *power*,
  zero and negative integers as powers,
  fractions as powers,

  and that you may well be aware of some of the *index laws*, these ideas will be re-visited in the first chapter of this text.

- **Function.**

It is assumed that the reader is familiar with the idea of a **function** being a rule that associates with each element in a set S a unique element from a set T. The set S is called the **domain** of the function, the set T is the **codomain** of the function and those elements of T that the function maps elements of S *onto* form the **range** of the function. If the domain of a function is not specifically stated then we assume it to be the set of all real numbers for which the function is defined. This is the **natural domain** of the function. If the function $f(x)$ maps the element a from the domain, onto the element b, of the range, we write $f(a) = b$.

The requirement that a function takes an element of the domain and maps it onto one, and only one, element of the range means that the graph of a function will pass the *vertical line test*. I.e. if a vertical line is moved from the left end of the x-axis to the right end of the x-axis it will not cut the graph at any more than one place at a time.

The reader should be familiar with the concept of polynomial functions in general and be especially familiar with linear, quadratic and cubic polynomials. This familiarity should include an understanding of the key features of the graphs of these particular types of functions, and of the reciprocal function. These key graphical features including

intercepts with the axes, turning points,
points of inflection, asymptotes,
concavity, symmetry
and the behaviour of the function as $x \to \pm\infty$.

- **Linear, quadratic and reciprocal functions in particular.**

Linear functions have:

Equations of the form $y = mx + c$.

Graphs that are straight lines with gradient m and cutting the y-axis at $(0, c)$.

Tables of values that have a *constant first difference* pattern.

Quadratic functions have

Equations of the form: $y = ax^2 + bx + c$.

Sometimes written as $y = a(x - p)^2 + q$

or $y = a(x - d)(x - e)$ each form allowing various key features of the graph of the quadratic function to be readily determined.

Graphs that are parabolic in shape with either a "hill", **a maximum**, or a "valley", **a minimum**.

Tables of values with a *constant second difference* pattern.

Reciprocal functions have

Equations of the form $y = \frac{k}{x}$, $x \neq 0$.

Graphs that are said to be **hyperbolic** in shape with the x and y axes as **asymptotes** to the curve.

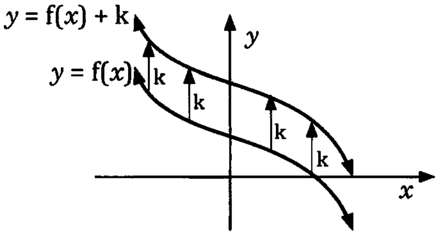
Table of values that have a constant product (equal to k).

- **Transformations.**
  The reader should be familiar with how the graphs of

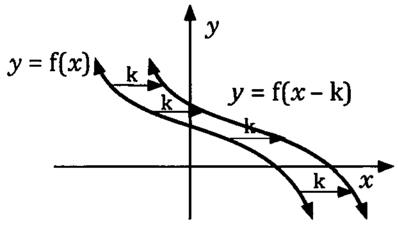
$$y = f(x) + k,$$
$$y = f(x - k),$$
$$y = af(x),$$
and
$$y = f(ax)$$

relate to that of $y = f(x)$.

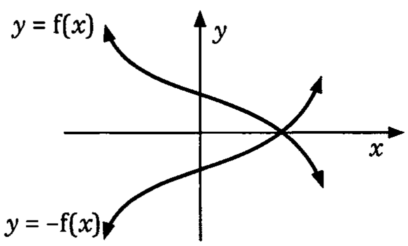
Adding k to the right hand side.
The graph of $y = f(x) + k$ will be that of $y = f(x)$ translated k units vertically upwards.
Thus if k is negative the translation will be vertically downwards.



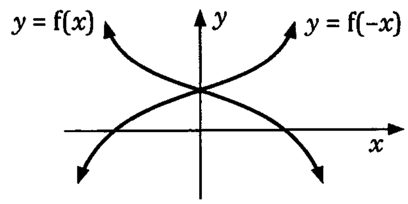
"Replacing x by $(x - k)$."
The graph of $y = f(x - k)$ will be that of $y = f(x)$ translated k units to the right.
Thus if k is negative the translation will be to the left.



"Multiplying the right hand side by −1."
The graph of $y = -f(x)$ will be that of $y = f(x)$ reflected in the x-axis.



"Replacing x by $-x$."
The graph of $y = f(-x)$ will be that of $y = f(x)$ reflected in the y-axis.



"Multiplying the right hand side by a."
The graph of $y = af(x)$ will be that of $y = f(x)$ dilated parallel to the y-axis with scale factor a. A point that is q units above the x-axis will be moved vertically to a point that is aq units above the x-axis. Points on the x-axis will not move.
If $a > 1$ the effect will be to stretch $y = f(x)$ vertically and if $0 < a < 1$ the effect will be to compress $y = f(x)$ vertically.

"Replacing x by ax."
The graph of $y = f(ax)$ will be that of $y = f(x)$ dilated parallel to the x-axis with scale factor $\frac{1}{a}$. A point that is p units from the y-axis will be moved horizontally to a point that is $\frac{p}{a}$ units from the y-axis. Points on the y-axis will not move.
If $a > 1$ the effect will be to compress $y = f(x)$ horizontally and if $0 < a < 1$ the effect will be to stretch $y = f(x)$ horizontally.

- **Equations.**
The reader should be able to solve linear equations, linear simultaneous equations and be familiar with factorisation, completing the square and the formula approach for solving quadratic equations. Familiarity with the ability of some calculators to solve equations is also assumed.

- **Coordinates.**
Whilst the reader may well be familiar with finding the length of a line joining two points and with determining the midpoint of a line joining two points, the most significant piece of information for this unit is the **gradient** of the line joining two points:
If a line passes through two points, A and B, then the gradient of the line is:

$$\frac{\text{the change in the } y\text{-coordinate in going from A to B}}{\text{the change in the } x\text{-coordinate in going from A to B}}$$

Thus the gradient of the straight line through A (x_1, y_1) and B (x_2, y_2) $= \frac{y_2 - y_1}{x_2 - x_1}$.

Note that in the previous formula, whilst $\frac{y_1 - y_2}{x_1 - x_2}$ would also give the correct answer, $\frac{y_1 - y_2}{x_2 - x_1}$ and $\frac{y_2 - y_1}{x_1 - x_2}$ would not.
Also remember that if two lines are perpendicular then the product of their gradients is -1. For example lines with gradients of 2 and $-\frac{1}{2}$ are perpendicular.

- **^nC_r.**
We use the notation ^nC_r for the number of combinations of r different objects taken from a set containing n different objects.

There are ^nC_r combinations of r objects chosen from n different objects where $^nC_r = \frac{n!}{(n-r)!\,r!}$.

Thus the number of combinations of three objects chosen from five different objects will be

$$\begin{aligned} {}^5C_3 &= \frac{5!}{(5-3)!\,3!} \\ &= \frac{5!}{2!\,3!} \\ &= \frac{5 \times 4}{2 \times 1} \\ &= 10 \end{aligned}$$

- **Expanding $(a + b)^n$.**

  The expansion of $(a + b)^n$ will be of the form

  $$k_0 a^n + k_1 a^{n-1} b^1 + k_2 a^{n-2} b^2 + k_3 a^{n-3} b^3 + \quad \ldots\ldots \quad + k_n b^n$$

  The first term is formed when we multiply the a from each bracket together to form a^n.

  For the second term we must choose one of the n brackets to supply the b and the others will then all supply an a. This can be done in ${}^nC_1$ ways.

  For the third term we must choose two of the n brackets each to supply b and the others will then each supply an a. This can be done in ${}^nC_2$ ways.

  Continuing with this approach leads to the binomial expansion:

  $$(a+b)^n = a^n + {}^nC_1 a^{n-1} b^1 + {}^nC_2 a^{n-2} b^2 + {}^nC_3 a^{n-3} b^3 + \ldots + {}^nC_n a^0 b^n$$

  this formula giving the same expansion for $(a + b)^2$, $(a + b)^3$, $(a + b)^4$ etc as we would obtain by multiplying out the brackets, or by obtaining the coefficients k_0, k_1, k_2 etc from the appropriate line of Pascal's triangle.

- **Factorising $a^n - b^n$.**

  You should be familiar with the fact that $a^2 - b^2 = (a - b)(a + b)$.

  However, probably not so familiar are the following facts:

  $$\begin{aligned} a^3 - b^3 &= (a-b)(a^2 + ab + b^2) \\ a^4 - b^4 &= (a-b)(a^3 + a^2b + ab^2 + b^3) \\ a^5 - b^5 &= (a-b)(a^4 + a^3b + a^2b^2 + ab^3 + b^4) \\ a^6 - b^6 &= (a-b)(a^5 + a^4b + a^3b^2 + a^2b^3 + ab^4 + b^5) \end{aligned}$$

  Check the validity of each of the above by expanding the right hand side in each case.

  To generalize:

  $$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + a^{n-4}b^3 + \quad \ldots \quad ab^{n-2} + b^{n-1})$$